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EXTENSION OF A LAYER WEAKENED BY TUNNELLING CUTS†

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The classical three-dimensional problem of the theory of elasticity for a layer weakened by generally curvilinear through-cuts is considered. A characteristic feature of the present study is that one-dimensional singular integral equations or, more precisely, an infinite system of such equations is used to solve the three-dimensional boundary-value problem. Numerical experiments indicate that the solution of this system by the reduction method converges quite rapidly almost everywhere in the range of variation of the thickness coordinate and the third approximation hardly increases the accuracy of results in this range. Hence the proposed procedure reduces the dimension of the problem by two. The accuracy of the solution needs to be improved in the immediate vicinity of the support of the layer, which involves singularities at the edge. This problem is not considered in this paper.

Problems of the above kind have been discussed more or less completely in [1–5]. An experimental study of the stressed state at the end of the edge reaching the support of the layer is presented in [6].

1. FORMULATION OF THE PROBLEM

Consider an elastic layer $-h \leqslant \bar{x}_3 \leqslant h, -\infty < x_1, x_2 < \infty$ weakened by cavity-like through-cuts tunnelling along the $\bar{x}_3$-axis whose cross-sections have the form of smooth open arcs L_j ($j = 1, 2, \ldots, k$). Suppose that the boundaries of the cavities are subject to a surface load $X_n^{\pm}(X_n^+ = X_n^- = X_n, n = 1, 2, 3)$. We shall assume that the curvature of the arcs and the functions X_n satisfy the Hölder condition on L_j, and X_n can be expanded in a Fourier series in $\bar{x}_3$ on $[-h, h]$. Below we consider the symmetric problem (with respect to the median plane of the layer) of the theory of elasticity.

We start from Lur'ye's homogeneous solutions [7], which we express as follows:
the biharmonic solution

$$u_1 - iu_2 = -2h\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z}\right)F, \quad u_3 = -h^2(\sigma - 1)\, x_3 \nabla^2 \varphi$$

$$\sigma_1 = 2\mu h \nabla^2 F, \quad \sigma_2 = 8\mu h \partial^2 F / \partial z^2, \quad \sigma_3 = 0, \quad z = x_1 + ix_2$$

$$F = (3\sigma - 1)\,\varphi + \frac{h^2}{2}(\sigma - 1)\left(\frac{1}{3} - x_3^2\right)\nabla^2\varphi, \quad \nabla^2\nabla^2\varphi = 0, \quad \bar{x}_3 = hx_3 \tag{1.1}$$

$$\partial_2^2 F_1 = -\partial_1^2 F_1 = 2\sigma\nabla^2\varphi, \quad \partial / \partial z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_i = \partial / \partial x_i, \quad \sigma = (1 - 2\nu)^{-1}$$

the vortex solution

$$u_1 - iu_2 = 4i\sigma h \sum_{m=1}^{\infty} \frac{\partial \varphi_m}{\partial z} \cos\rho_m x_3, \quad u_3 = 0$$

$$\sigma_1 = 0, \quad \sigma_2 = -16i\mu\sigma h \sum_{m=1}^{\infty} \frac{\partial^2 \varphi_m}{\partial z^2} \cos\rho_m x_3, \quad \sigma_{33} = 0 \tag{1.2}$$

$$\sigma_3 = -4i\mu\sigma \sum_{m=1}^{\infty} \rho_m \frac{\partial \varphi_m}{\partial z} \sin\rho_m x_3$$

$$\rho_m = \pi m, \quad \gamma_m = \rho_m / h, \quad (\nabla^2 - \gamma_m^2)\,\varphi_m = 0$$

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the potential solution

$$\begin{aligned}
&u_1 - iu_2 = 2h\frac{\partial}{\partial z}\operatorname{Re}\sum_{k=1}^{\infty}\alpha_k(x_3), \psi_k, \quad u_3 = \operatorname{Re}\sum_{k=1}^{\infty}\mu_k(x_3)\,\psi_k \\
&\sigma_1 = \frac{2\mu}{h}\operatorname{Re}\sum_{k=1}^{\infty}\beta_k(x_3)\,\psi_k, \quad \sigma_2 = -8\mu h\frac{\partial^2}{\partial z^2}\operatorname{Re}\sum_{k=1}^{\infty}\alpha_k(x_3)\,\psi_k \\
&\sigma_3 = 4\mu\sigma\frac{\partial}{\partial z}\operatorname{Re}\sum_{k=1}^{\infty}\kappa_k(x_3)\,\psi_k, \quad \sigma_{33} = \frac{2\mu\sigma}{h}\operatorname{Re}\sum_{k=1}^{\infty}\nu_k(x_3)\,\psi_k \\
&\alpha_k(x_3) = (t_k/\delta_k - \sigma\tau_k)\cos\delta_k x_3 - \sigma x_3 t_k\sin\delta_k x_3 \\
&t_k = \sin\delta_k, \quad \tau_k = \cos\delta_k, \quad \operatorname{Re}\delta_k > 0, \quad \operatorname{Im}\delta_k > 0 \qquad (1.3) \\
&\beta_k(x_3) = ((2\sigma - 1)\,\delta_k t_k - \sigma\delta_k^2\tau_k)\cos\delta_k x_3 - \sigma\delta_k^2 x_3 t_k\sin\delta_k x_3 \\
&\kappa_k(x_3) = \delta_k(\tau_k\sin\delta_k x_3 - x_3 t_k\cos\delta_k x_3)), \quad \mu_k(x_3) = 2\sigma\kappa_3(x_3) - \alpha_k(x_3), \\
&\nu_k(x_3) = (\delta_k t_k + \delta_k^2\tau_k)\cos\delta_k x_3 + x_3\delta_k^2 t_k\sin\delta_k x_3 \\
&\lambda_k = \delta_k/h, \quad (\nabla^2 - \lambda_k^2)\,\psi_k = 0
\end{aligned}$$

The integral representations of the functions in (1.1)–(1.3) must ensure the existence of displacement jumps and the continuity of the stress vector as L_j ($j = 1, 2, \ldots, k$) are crossed, as well as the decay of displacements and stresses at infinity. Below we shall be concerned with the construction of such representations correct in the above sense.

We set

$$\begin{aligned}
&\varphi(z) = \operatorname{Re}\int_L\left(p(\zeta)\frac{\partial G}{\partial\zeta} + p_*(\zeta)\frac{\partial}{\partial\zeta}\nabla^2 G\right)d\zeta + \int_L q(\zeta)\,\nabla^2 G\,ds \\
&F_1(z) = -4\sigma\operatorname{Re}\int_L p(\zeta)\,[\ln(\zeta - z) - 1 - \ln h]\,(\zeta - z)\,d\zeta \\
&\psi_k(z) = \int_L q_k(\zeta)\,K_0(\lambda_k r)\,ds + \int_L\left(p_k(\zeta)\frac{\partial}{\partial\zeta}K_0(\lambda_k r)\,d\zeta + p_k^*(\zeta)\frac{\partial}{\partial\zeta}K_0(\lambda_k r)\,d\bar{\zeta}\right) \qquad (1.4) \\
&\varphi_m(z) = \int_L q_m^*(\zeta)\,K_0(\gamma_m r)\,ds + 2\operatorname{Re}\int_L R_m(\zeta)K_0(\gamma_m r)\,d\zeta \\
&G = r^2\ln\frac{r}{h}, \quad r = |\zeta - z|, \quad \zeta = \xi_1 + i\xi_2 \in L, \quad \operatorname{Im} q_m^*(\zeta) = 0
\end{aligned}$$

The functions $p(\zeta) = \{p_j(\zeta), \zeta \in L_j\}, \ldots, R_m(\zeta) = \{R_{mj}(\zeta), \zeta \in L_j\}$ are to be determined from the boundary conditions, but first they must be expressed in terms of the displacement jumps on L_j.

We expand all even components of the displacement vector, the stress tensor and the external load in Fourier series of the form $u = \Sigma\, u^{(m)}\cos\rho_m x_3$ and all odd components in series of the form $U = \Sigma U^{(m)}\sin\rho_m x_3$. Singularities of the form $(\zeta - z)^{-3}$ appear in the kernels of the integral representations when the Fourier coefficients $\sigma_{ij}^{(m)}$ are determined from (1.4). To remove these singularities we introduce relationships between the densities in the representations (1.4)

$$\begin{aligned}
&4(1 - 3\sigma)\,p_*(\zeta) = \sum_{k=1}^{\infty}(\alpha_k^{(0)}p_k(\zeta) + \bar{\alpha}_k^{(0)}\overline{p_k^*(\zeta)}) \\
&\frac{8(-1)^m(\sigma - 1)\,h^2}{\pi^2 m^2}\,p(\zeta) - 4i\sigma R_m(\zeta) = \sum_{k=1}^{\infty}(\alpha_k^{(m)}p_k(\zeta) + \overline{\alpha_k^{(m)}}\,\overline{p_k^*(\zeta)}) \qquad (1.5)
\end{aligned}$$

The boundary conditions on L have the form

$$\{\sigma_1^{(m)} - e^{2i\psi}\sigma_2^{(m)}\}^{\pm} = \pm 2e^{i\psi}\{X_1^{(m)} - iX_2^{(m)}\}^{\pm}, \quad m = 0, 1, \ldots$$

$$\operatorname{Re}\{e^{i\psi}(\sigma_3^{(m)})^{\pm}\} = \pm(X_3^{(m)})^{\pm}, \quad m = 1, 2, \dots \tag{1.6}$$

$$\sigma_1 = \sigma_{11} + \sigma_{22}, \quad \sigma_2 = \sigma_{22} - \sigma_{11} + 2i\sigma_{12}, \quad \sigma_3 = \sigma_{13} - i\sigma_{23}$$

The upper sign corresponds to the left edge of the cut as one moves from the tip a_j to b_j and ψ is the angle between the normal line to the left edge and the Ox_1 axis (Fig. 1).

The requirement that the stress vector should be continuous as the cut is crossed (in this case it is sufficient for the boundary equalities (1.6) to be satisfied on one of the cut edges) and the requirement that the displacement vector should have discontinuities on L lead to the relations

$$q_m^*(\zeta) = -\frac{[u_3^{(m)}]}{4\pi\sigma h}, \quad q(\zeta) = [4(1-3\sigma)]^{-1} \operatorname{Re} \sum_{k=1}^{\infty} \alpha_k^{(0)} q_k(\zeta)$$

$$p(\zeta) = -\frac{U^{(0)} + iV^{(0)}}{8\pi\sigma h}, \quad U^{(m)} = \frac{d[u_n^{(m)}]}{ds} - \frac{[u_s^{(m)}]}{\rho} \tag{1.7}$$

$$V^{(m)} = \frac{d[u_s^{(m)}]}{ds} + \frac{[u_n^{(m)}]}{\rho}, \quad R_m = \frac{ih(U^{(m)} + iV^{(m)})}{2\sigma\pi^3 m^2}$$

according to which the densities $q_m^*(\zeta), p(\zeta), R_m(\zeta)$ can be directly expressed in terms of the displacement jumps along the cuts. The requirements also lead to three pairs of infinite systems of linear algebraic equations relating the remaining densities to the jumps.

Introducing the representations

$$q_k(\zeta) = \frac{1}{2\pi h} \sum_{j=1}^{\infty} q_{kj}[u_n^{(j)}] \tag{1.8}$$

$$p_k + p_k^* = \frac{2h}{\pi^2} \sum_{j=1}^{\infty} \left\{ q_{kj} \frac{1}{\pi j^2} \left(U^{(j)} - (-1)^j \frac{\sigma - 1}{2\sigma} U^{(0)} \right) + \frac{q_{kj}^*}{2\sigma j} U^{(j)} \right\} \tag{1.9}$$

$$p_k - p_k^* = \frac{1}{\pi} \sum_{j=1}^{\infty} \left\{ \frac{h}{\pi\sigma j^2} S_{kj} V^{(j)} + S_{kj}^* \left(\frac{[u_3^{(j)}]}{j} - \frac{(-1)^j h(\sigma - 1)}{\pi\sigma j^2} V^{(0)} \right) \right\} \tag{1.10}$$

we obtain the following "standard" systems ($m, j = 1, 2, \dots$; summation is over k from one to infinity)

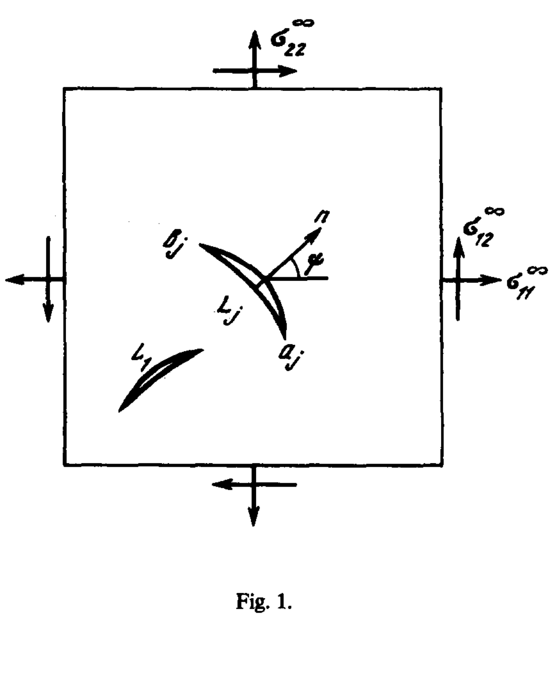

Fig. 1.

$$\mathrm{Re}\sum \alpha_k^{(m)} q_{kj} = \delta_{mj}, \quad \mathrm{Re}\sum \kappa_k^{(m)} q_{kj} = 0 \tag{1.11}$$

$$\mathrm{Re}\sum \alpha_k^{(m)} q_{kj}^* = 0, \quad \mathrm{Re}\sum \kappa_k^{(m)} q_{kj}^* = -\delta_{mj} \tag{1.12}$$

$$\mathrm{Im}\sum \Lambda_k^{(m)} S_{kj} = -\delta_{mj}, \quad \mathrm{Im}\sum \mu_k^{(m)} S_{kj} = 0 \tag{1.13}$$

$$\mathrm{Im}\sum \Lambda_k^{(m)} S_{kj}^* = 0, \quad \mathrm{Im}\sum \mu_k^{(m)} S_{kj}^* = \delta_{mj} \tag{1.14}$$

where q_{kj}, q_{kj}^*, S_{kj}, S_{kj}^* are to be determined and δ_{mj} is the Kronecker delta.

The resulting systems admit of an exact solution. We consider (1.11). Multiplying the rows in the first system by $\cos \rho_m x_3$ and in the second system by $\sin \rho_m x_3$ and summing the results over m, we find that

$$\sum q_{kj}(\alpha_k(x_3) - \alpha_k^{(0)}) = f_j, \quad \sum q_{kj}\kappa_k(x_3) = 0$$
$$f_j = 2\cos\rho_j x_3, \quad \mathrm{Re}\,\delta_k > 0 \tag{1.15}$$

The functions $\alpha_k(x_3)$, $\mu_k(x_3)$ constitute a solution of the following (non-self-adjoint) boundary-value problem

$$\begin{gathered}\alpha_3''(x_3) + (1+\sigma)\,\delta_k^2\alpha_k(x_3) + \sigma\mu_k'(x_3) = 0\\ (1+\sigma)\,\mu_k''(x_3) + \delta_k^2\mu_k(x_3) + \sigma\delta_k^2\alpha_k'(x_3) = 0\\ \alpha_k'(\pm 1) + \mu_k(\pm 1) = 0, \quad (\sigma-1)\,\delta_k^2\alpha_k(\pm 1) + (\sigma+1)\,\mu_k'(\pm 1) = 0\end{gathered} \tag{1.16}$$

Using (1.16), one can reduce the functional equations (1.15) to the following equivalent form

$$\sum q_{kj}Y_k(x_3) = -\frac{4\sigma}{\sigma+1} f_j''(x_3), \quad \sum q_{kj}\delta_k^2 Y_k(x_3) = \frac{8\sigma}{\sigma+1} f_j^{(4)}(x_3) \tag{1.17}$$

Here Y_k is the solution of the boundary-value problem

$$Y_k^{(4)} + 2\delta_k^2 Y_k'' + \delta_k^4 Y_k = 0, \quad Y_k(\pm 1) = Y_k'(\pm 1) = 0 \tag{1.18}$$

The generalized orthogonality condition has the form

$$\int_{-1}^{1} \{2Y_k'Y_m' - (\delta_k^2 + \delta_m^2)\,Y_k Y_m\}\,dx_3 = 0, \quad m \neq k \tag{1.19}$$

Using (1.18) and (1.19), and the scheme developed in [8], we obtain

$$q_{kj} = \frac{2\sigma\delta_k^2}{\sigma+1}\int_{-1}^{1} f_j''(x_3)\,Y_k(x_3)\,dx_3 \Big/ \int_{-1}^{1}\{(Y_k')^2 - \delta_k^2 Y_k\}\,dx_3$$

Computing the above integrals, we finally obtain

$$q_{kj} = (-1)^{j+1}\frac{4\pi^2 j^2\delta_k^2}{(\sigma+1)\,(l_{kj}\tau_k)^2}, \quad l_{kj} = \delta_k^2 - \rho_j^2 \tag{1.20}$$

In a similar way we can find the solutions of the "standard" systems (1.12)–(1.14)

$$q_{kj}^* = (-1)^{j+1}\frac{2\pi j}{(l_{kj}\tau_k)^2}\left(\pi^2 j^2 - \frac{3\sigma+1}{\sigma+1}\delta_k^2\right)$$
$$S_{kj} = i(2\sigma q_{kj} + \pi j q_{kj}^*), \quad S_{kj}^* = i q_{kj} \tag{1.21}$$

We have therefore established that the infinite systems of equations (1.11)–(1.14) are solvable. The solutions are obtained in closed form and are given by (1.20) and (1.21), all the densities in the integral representations (1.4) being expressed in terms of physical quantities, namely, the "displacement" jumps on L.

Formulae (1.9) and (1.10) can be simplified considerably by substituting into them the values of the coefficients from (1.20) and (1.21) and taking the sums of certain series. We have

$$p_k + p_k^* = -\frac{h}{\pi^3\sigma}\left\{\mathrm{E}_k U^{(0)} + i\sum_{j=1}^{\infty}\frac{S_{kj}}{j^2}U^{(j)}\right\},$$

$$p_k - p_k^* = -\frac{h}{\sigma\pi^3}\left\{i\mathrm{E}_k V^{(0)} - \sum_{j=1}^{\infty}\frac{S_{kj}}{j^2}V^{(j)} - \frac{i\pi\sigma}{h}\sum_{j=1}^{\infty}\frac{q_{kj}}{j}[u_3^{(j)}]\right\},\quad \mathrm{E}_k = \frac{2(\sigma-1)\pi^2}{(\sigma+1)\delta_k^2\tau_k^2} \tag{1.22}$$

Note that representations (1.4) also remain valid for the second fundamental problem, for example, in the presence of a rigid insertion in a cavity. However in the latter case the densities will be expressed in terms of the jumps of the stress vector on L.

2. INTEGRAL EQUATIONS OF BOUNDARY-VALUE PROBLEM (1.6)

We obtain integral representations for the stresses σ_{ij} by substituting the formulae from (1.4) for the functions in (1.1)–(1.3). Expanding the resulting expressions in Fourier series in x_3, we obtain integral representations of the Fourier coefficients $\sigma_{ij}^{(m)}$. Then, by satisfying the boundary conditions (1.6) on one edge of L and taking (1.5), (1.7), (1.8), and (1.22) into account, we arrive at an infinite system of one-dimensional singular integro-differential equations of the first kind

$$\int_L X^{(0)}(\zeta)\,g(\zeta,\zeta_0)\,ds + \ldots = \frac{4\pi\sigma}{3\sigma-1}F_0(\zeta_0)$$

$$\int_L\left(\frac{2\sigma}{\pi}X^{(m)}(\zeta) - im(\sigma-1)\,\omega_3^{(m)}(\zeta)\right)g(\zeta,\zeta_0)\,ds + \ldots = \tag{2.1}$$

$$= 2(\sigma+1)\,F_m(\zeta_0) + \frac{4(-1)^m(\sigma-1)^2}{3\sigma-1}F_0(\zeta_0)$$

$$\int_L\left(\frac{h}{\pi}\frac{d\omega_3^{(m)}}{ds} - m\omega_4^{(m)}\right)g(\zeta,\zeta_0)\,ds + \ldots = 4F_m^*(\zeta_0),\quad m = 1,2,\ldots$$

Here

$$\mu F_0(\zeta) = N^{(0)} - iT^{(0)} - \frac{1}{2}\{\sigma_{11}^\infty + \sigma_{22}^* - e^{2i\psi}(\sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty)\}$$

$$\mu F_m(\zeta) = N^{(m)} - iT^{(m)},\quad 2\mu F_m^*(\zeta) = X_3^{(m)},\quad N^{(m)} - iT^{(m)} = e^{i\psi}(X_1^{(m)} - iX_2^{(m)})$$

$$X^{(m)}(\zeta) = U^{(m)} - iV^{(m)},\quad U^{(m)} = \omega_1^{(m)}(\zeta),\quad V^{(m)} = \omega_2^{(m)}(\zeta),\quad m = 0,1,\ldots$$

$$h\omega_3^{(m)}(\zeta) = [u_3^{(m)}],\quad h\omega_4^{(m)}(\zeta) = [u_s^{(m)}],\quad g(\zeta,\zeta_0) = \mathrm{Im}\,\frac{e^{i\psi}}{\zeta-\zeta_0}$$

The terms with regular kernels are not written down in (2.1). The structure of the system is such that all unknowns are "tied up" in its regular part, the characteristic part of the system containing exactly three functions $[u_n^{(m)}]$, $[u_s^{(m)}]$, $[u_3^{(m)}]$ for each fixed $m = 1, 2, \ldots$ and two functions $[u_n^{(0)}]$ and $[u_s^{(0)}]$ for $m = 0$.

Since the displacement jumps vanish at the tips of L_j, the system of equations (2.1) must be considered together with the additional conditions

$$\int_{L_j}(U^{(m)} + iV^{(m)})\,d\zeta = 0,\quad \int_{L_j} d[u_3^{(m)}] = 0,\quad j = 1,2,\ldots,k,\ m = 0,1,\ldots \tag{2.2}$$

and the functions $U^{(m)}$, $V^{(m)}$ and $d[u_3^{(m)}]/ds$ must be sought in the class h_0 [9].

Let us consider more carefully the singular part of (2.1). For simplicity, we assume that L is the section $x_2 = 0, -l \leqslant x_1 \leqslant l$. Then we have the following equations:
for $m = 0$

$$\int_{-l}^{l} \frac{d[u_2^{(0)} + iu_1^{(0)}]}{x - x_0} = N_0(x_0), \quad -l < x_0 < l \tag{2.3}$$

for $m = 1, 2, \ldots$

$$\int_{-l}^{l} \frac{d[u_2^{(m)}]}{x - x_0} = N_m(x_0), \quad -l < x_0 < l \tag{2.4}$$

$$\int_{-l}^{l} y_{im}(x) \frac{dx}{x - x_0} = N_{im}(x_0), \quad i = 1, 2 \tag{2.5}$$

Here

$$y_{1m}(x) = \frac{d}{dx}[u_1^{(m)}] + \alpha_m [u_3^{(m)}], \quad \alpha_m = \frac{\sigma - 1}{2\sigma}\gamma_m$$

$$y_{2m}(x) = \frac{d}{dx}[u_3^{(m)}] - \gamma_m [u_1^{(m)}]$$

The functions $N_m(x)$ and $N_{im}(x)$ are unknown.

Equations (2.3) and (2.4) are solvable, their solutions being fixed by additional conditions of type (2.2). By the substitution

$$\omega_{1m} = \frac{d}{dx}[u_1^{(m)}], \quad \omega_{2m} = \frac{d}{dx}[u_3^{(m)}]$$

we reduce the remaining system (2.5) to the standard form [9, 10]

$$\int_{-l}^{l} \frac{\omega_{1m} dx}{x - x_0} + \alpha_m \int_{-l}^{l} \omega_{2m} \ln\left|\frac{l - x_0}{x - x_0}\right| dx = N_{1m}(x_0), \quad m = 1, 2, \ldots$$

$$\int_{-l}^{l} \frac{\omega_{2m} dx}{x - x_0} - \gamma_m \int_{-l}^{l} \omega_{1m} \ln\left|\frac{l - x_0}{x - x_0}\right| dx = N_{2m}(x_0)$$

where the kernels in the second terms are now regular. It follows that the characteristic part of (2.1) is solvable in h_0 for any fixed $m = 0, 1, \ldots$.

3. STRESS INTENSITY FACTORS

We introduce a parameterization $\zeta = \zeta(\delta)$, $\zeta_0 = \zeta(\delta_0)$, $-1 \leqslant \delta, \delta_0 \leqslant 1$ of L_j (the subscript j will be omitted below). Correspondingly, we set

$$\omega_p^{(m)}(\zeta) = \frac{\Omega_p^{(m)}(\delta)}{s'(\delta)\sqrt{1 - \delta^2}}, \quad p = 1, 2; \quad m = 0, 1, \ldots$$

$$\frac{d\omega_3^{(m)}}{ds} = \frac{\Omega_3^{(m)}(\delta)}{s'(\delta)\sqrt{1 - \delta^2}}, \quad s'(\delta) = \frac{ds}{d\delta} > 0 \quad (\Omega_p^{(m)}(\delta) \in H[-1, 1])$$

Using these expressions, formulae (1.7), (1.8) and (1.12) relating the densities in the integral representations (1.4) with the displacement jumps, as well as (1.1)–(1.3), as a result of a detailed asymptotic analysis of the integral representations for the stresses we find that

$$K_{\rm I} - iK_{\rm II} = -\frac{\mu\sigma}{\sigma+1}\sqrt{\frac{\pi}{s'(\mp 1)}}\sum_{m=0}^{\infty}(\Omega_1^{(m)}(\mp 1) - i\Omega_2^{(m)}(\mp 1))\cos m\pi x_3$$

$$K_{\rm III} = -\frac{\mu h}{2}\sqrt{\frac{\pi}{s'(\mp 1)}}\sum_{m=1}^{\infty}\Omega_3^{(m)}(\mp 1)\sin m\pi x_3$$

$$K_{\rm I} = \sqrt{2\pi r}\,\sigma_n,\quad K_{\rm II} = \sqrt{2\pi r}\,\sigma_{ns},\quad K_{\rm III} = \sqrt{2\pi r}\,\sigma_{n3}\quad (r \to 0)$$

Here σ_n, σ_{ns} and σ_{n3} are the normal and tangential stresses in the area beyond the crack tip, the upper sign corresponding to the beginning a of the crack and the lower one to the end b (Fig. 1).

It follows that the stress intensity factors can be expressed in a natural way in terms of the functions

$$U = \frac{d}{ds}[u_n(\zeta, x_3)] - \frac{1}{\rho}[u_s(\zeta, x_3)],\quad \frac{d}{ds}[u_3(\zeta, x_3)]$$

$$V = \frac{d}{ds}[u_s(\zeta, x_3)] + \frac{1}{\rho}[u_n(\zeta, x_3)],\quad (\zeta = a_j \vee b_j, x_3 \in [-1, 1])$$

4. SOME NUMERICAL RESULTS

As an example we consider a layer weakened by a tunnelling parabolic cut $\xi_1 = p_1\delta$, $\xi_2 = p_2\delta^2$ $(-1 \leqslant \delta \leqslant 1)$ subject to a uniform field σ_{ij}^{∞}. A load X_n $(n = 1, 2, 3)$ is assumed on the surfaces of the cut.

In the numerical realization of the algorithm the system of integral equations (2.1) was reduced by the mechanical quadratures method [11] to a linear system of algebraic equations, which was solved by the reduction method. In the Nth approximation the first $3N + 2$ real equations and, correspondingly, the $3N + 2$ unknowns $[u_1^{(0)}]$, $[u_2^{(0)}]$, $[u_1^{(m)}]$, $[u_2^{(m)}]$, $[u_3^{(m)}]$ $(m = 1, 2, \ldots, N)$ were retained in the system. Two integral equations and, correspondingly, the two densities $[u_1^{(0)}]$, and $[u_2^{(0)}]$ were retained in the zeroth approximation. Computations were carried out for $N = 0, 1, 2, 3, 4$. The third approximation introduces practically no improvement in the accuracy of results for $K_{\rm I}$ and $K_{\rm II}$ in the range $|x_3| < 0.9$. Even more rapid convergence was observed for $K_{\rm III}$ in the range $|x_3| \leqslant 1$.

Let $\sigma_{22}^{\infty} \neq 0$, $\sigma_{11}^{\infty}\ \sigma_{12}^{\infty} = X_n = 0$ $(n = 1, 2, 3)$. In Fig. 2 we show diagrams of the distribution of the specific stress intensity factor $\langle K_{\rm I}\rangle = K_{\rm I}/(\sigma_{22}^{\infty}\sqrt{(\pi l)})$ with respect to the thickness coordinate for various p_2 and h/l ($2l$ being the length of the crack). The upper three graphs correspond to a straight line crack ($p_2 = 0$) and the lower ones to a parabolic crack ($p_2 = 0.5$). When $p_2 = 1$ these three curves are very close to one another and to $\langle K_{\rm I}\rangle = 0.2$. Here and henceforth $p_1 = 1$ and $\nu = 0.3$ were taken in the computations.

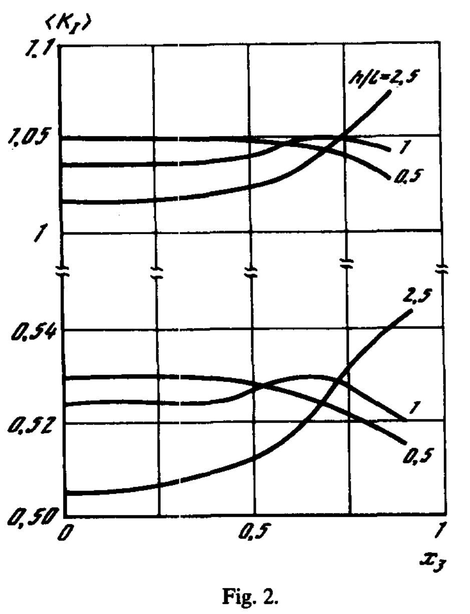

Fig. 2.

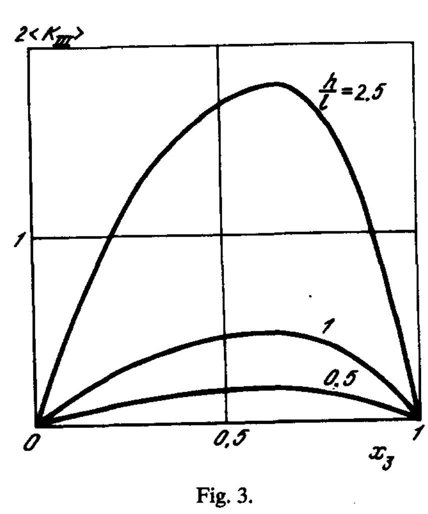

Fig. 3.

Now let $\sigma_{12}^{\infty} \neq 0$ and $\sigma_{11}^{\infty} = \sigma_{22}^{\infty} = X_n = 0$. In this case $\langle K_{II}\rangle$ is practically independent of x_3. The values of $\langle K_{II}\rangle$ for a straight crack for $h/l = 2.5$ and $h/l = 1$ vary from 1.020 to 1.013 and from 1.030 to 1.024, respectively. For a parabolic crack the values of $\langle K_{II}\rangle$ are approximately the same when $h/l = 0.5, 1, 2.5$. When $p_2 = 0.5$, we have $0.47 < \langle K_{II}\rangle <$ 0.48 for $p_2 = 1$, $0.08 < \langle K_{II}\rangle < 0.10$.

We also consider the case when $\sigma_{ij}^{\infty} = 0$ and shear forces $X_3 = X_3^{(1)} \sin \pi x_3$, $X_1 = X_2 = 0$ act on the surface of a rectilinear crack. The graphs of $\langle K_{III}\rangle = K_{III}(X_3^{(1)} \sqrt{(\pi l)})^{-1}$ are presented in Fig. 3.

The procedure proposed in this paper for solving the first boundary-value problem of the theory of elasticity with tunnelling through-cuts can be used without any major modifications to solve the second boundary-value problem, for example, for a layer with thin curvilinear rigid inclusions. It can also be extended to the case of a multiply connected finite cylinder containing both cracks and perforation cavities.

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